Examination Languages and Machines
10 juli 2014, 14.00-17.00 (translated)

Duration 3 hours. Closed book. You are allowed to use theorems from the Lecture Notes, if you phrase them correctly. Give clear and crisp arguments for all your assertions. Write legible.

Write your name on all pages you turn in; number them.

Exercise 1 (10 %). Consider a language L over the alphabet Σ. Fill in the dots (...) with a language property defined in the course.

(a) L is ... $\equiv$ $\exists\, M : L = L(M)$ and M is a Turing Machine that terminates for every input.

(b) L is ... $\equiv$ $\exists\, M : L = L(M)$ and M is a nondeterminic finite state machine.

(c) L is ... $\equiv$ $\exists\, M : L = L(M)$ and M is a pushdown machine.

(d) L is ... $\equiv$ $\exists\, M : L = L(M)$ and M is a linearly bounded machine.

(e) Give all valid implications between these four assertions about L.

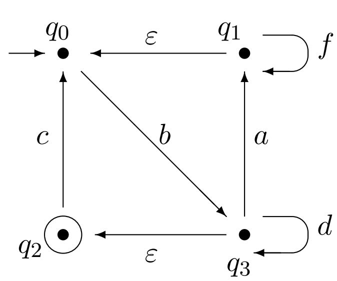
Exercise 2 (12 %). When is a context-free grammar $G = (V, \Sigma, P, S)$ called *essentially noncontracting*? When is it called *productive*? Give the two definitions.

(b) Let the context-free grammar G be given by the production rules:

$$\begin{array}{lcl} S & \to & DD \mid aDb \\ D & \to & Sc \mid \varepsilon \ . \end{array}$$

Use the standard algorithm to determine an equivalent *productive* grammar. Give all intermediate results.

Exercise 3 (10 %). Consider the alphabet $\Sigma = \{a, b, c, d, f\}$ and the nondeterministic finite state machine M with ε-tranitions, with the state diagram:



Use the standard algorithm to determine a regular expression for the language $L(M)$.

Exercise 4 (16 %). (a) Phrase the Pumping Lemma for *regular* languages.

(b) Given is the language $L_4 = \{a^p b^q \mid p \neq q\}$ over the alphabet $\Sigma = \{a, b\}$. Prove that this language is not regular. For this purpose, you can use the closure properties of regular languages and the Pumping Lemma (for a language different from L_4).

Exercise 5 (10 %). Consider the language L_5 over $\Sigma = \{a, b, c\}$ generated by the context-free grammar

$$\begin{aligned} S &\to \varepsilon \mid aFF \\ F &\to b \mid cS \ . \end{aligned}$$

Construct a simple pushdown machine that accepts the language L_5. It suffices to give the state diagram, and to make it clear why this pushdown machine accepts the language L_5.

Exercise 6 (12 %). Consider the alphabet $\Sigma = \{a\}$ and the language

$$L_6 = \{a^n \mid \exists\, k \in \mathbb{N} : n = 3^k\} \ .$$

Construct an always terminating Turing machine M with $L(M) = L_6$. Give the complete state diagram. Indicate in which states the computation can terminate when the input does not belong to L_6. Clearly indicate how many tapes (or tracks) you are using, whether the machine is deterministic or nondeterministic, why the machine always terminates, and why it accepts the language L_6.

Exercise 7 (16 %). (a) Give the definitions of *decidable* and *semi-decidable* languages.

(b) Let L be a semi-decidable language, and let L' be a decidable language. Prove that the difference set $L \setminus L' = \{w \in L \mid w \notin L'\}$ is semi-decidable.

(c) Let L and L' be two semi-decidable languages. Does this imply that the difference set $L \setminus L'$ is semi-decidable? Give good arguments for your answer.

(d) Does there exist a triple of languages L_1, L_2, and L_3 with $L_1 \subseteq L_2 \subseteq L_3$ such that L_1 is context-free, that L_2 is not decidable, and that L_3 is regular? If so, give an example. If not, why not?

Exercise 8 (14 %). The Lecture Notes describe a way to encode a Turing machine $M \in \mathit{TM0}$ by means of a string $R(M)$ over the alphabet $\mathbb{B}$, and a universal Turing machine that can be used to simulate any Turing machine M thus encoded.

Now consider the language

$$\begin{aligned} L_8 = \{R(M)1^n \mid M \in \mathit{TM0}, n \in \mathbb{N} :\ & \text{execution of machine } M \\ & \text{with empty input } (\varepsilon) \text{ terminates within } n \text{ steps } \}. \end{aligned}$$

(a) Prove that the language L_8 is decidable.
(b) The language L'_8 consists of the initial segments of L_8, i.e.,

$$L'_8 = \{u \mid \exists\, v : uv \in L_8\} \ .$$

Prove that L'_8 is not decidable.